Ratios

Goals for this chapter

1. Learn what a ratio is
2. Learn to deal with ratios algebraically

What is a Ratio?

A **ratio** is a way of expressing a certain kind of relationship between two values; it is a certain type of comparison between two values. So far this isn't terribly informative, but it should sound familiar. What else that we've been talking about is a comparison between two values?

That's right – an equation. Both ratios and equations are comparisons between values, and, in fact, ratios can always be expressed, or written, as equations. In fact, not only *can* they be written as equations, we always *should* write ratios as equations. (This is the first lesson of this chapter. Learn it well). Why should we always write ratios as equations? Well, we just spent a couple of chapters learning how to deal with equations; we have this tool, algebra, that we understand and are good at using, so we should use it whenever possible.

You'll notice that this – using algebra when we can – is a theme throughout this book. There's a saying, "When you have a hammer, everything looks like a nail." One of my old law school teachers used to use it often to criticize lawyers for wanting to sue everyone all the time. But, in our case, we should change the saying. "If you have a hammer, and you can fix something by hitting it, there's no reason to go to Home Depot." In other words, once we have a tool that's can be applicable to a situation, we should use it rather than learning something new. Our hammer is algebra, and we are going to use it when and wherever possible. You'll see that ratios are algebra, geometry is algebra, probability is algebra… Ahh, the mighty hammer of algebra.

A ratio is a special type of comparison. It tells you "for every one (or two or three…) of these, you always have one (or two or three…) of those." Imagine you are cooking a recipe. The recipe tells you to use two cups flour and one cup butter. But you want to make three as much food (you've got a big party coming up). What do you do? You use three times as much of each thing, right? Of course you do. Which means that now you have six cups of flour and one cup of butter. Notice that in either case – under the original recipe or when you cook three times as much – there is the same relationship between flour and butter. There will *always* be twice as much flour as butter in the recipe. This is a ratio.

In a ratio, the number of one thing has to be something times the number of the other thing. In the flour/butter ratio, flour is two times butter. The number doesn't have to be a whole number, though. You might have one thing being 3/4 as much as another thing. In a ratio, the relationship between one thing and another is always based on multiplication (or division, the opposite of multiplication).

The following is not a ratio: I always have three more cookies than you do. It isn't a ratio because the relationship is based on addition, not multiplication. Imagine that you have three cookies. I have six, twice as many. Now double your cookies, so you have six. I will have to have nine, which is only 3/2 as many. The only relationship between you and me that stays the same is the one of addition (me having three more), not one of multiplication (me having twice as many, or me having 3/2 as many). That is why this is not a ratio.

Now, so far this may not look much different than what we've done in the last few chapters. If we were to write an equation describing the flour and butter situation above, it might look like

$$\text{flour} = 2(\text{butter})$$

Let's fiddle with this for a second. Move the "butter" to the same side as the flour. This requires dividing both sides by "butter".

$$\frac{\text{flour}}{\text{butter}} = 2$$

Now, this is how ratios are normally written, as fractions. Ratios can also be written with colons. For example, "the ratio of flour to butter" can be written

$$\text{flour} : \text{butter}$$

The GMAT will occasionally write ratios in this way. You should never do so. Always write ratios as fractions. Why? Because we don't know how to use colons in equations. If you were given the following equation, what would you do?

$$x : y = \frac{3}{2}$$

I don't know either. Colons are useless in algebra, and in almost every ratio question you will end up doing algebra. If you had written the above as

$$\frac{x}{y} = \frac{3}{2}$$

you would know how to deal with this. So we always write ratios as fractions. In addition, every ratio should be part of an equation. Often students will be told "the ratio of a to b is 1 to 5" and they'll write down

$$\frac{1}{5}$$

This is useless. 1/5 is what? How would we combine this with other equations? We can't. 1/5 is the ratio of a to b. Remember, "is" means equals. So 1/5 equals "of a to b". In ratios, we put the "of" on the top of the fraction, and the "to" on the bottom.

$$\frac{1}{5} = \frac{a}{b}$$

Now, *that* is something we can use algebra on.

Let's return to our discussion of why ratios are important. We've seen that they are useful in recipes. In fact, in *any* process where ingredients must be mixed in certain relationships called "proportions"[1]) we use ratios – cooking, bartending, chemistry. Ratios are also useful in business: if you want to maintain a majority share in a company, for example, the ratio of your shares to the total number of shares must be greater than 1/2. That is:

$$\frac{\text{your shares}}{\text{total shares}} > \frac{1}{2}$$

It is very difficult to express this relationship using only addition. Ratios are one of the most useful mathematical concepts in every day life. As we will see later, all percents are ratios, and we use percents every day.

Ratios in Equations

We now understand what ratios are and why they are important. Lets take a little time to practice writing equations with them.

Example 1:

The ratio of men to women in a certain class is 3 to 1.

As I said before, in ratios, "of" goes on top of the fraction, and "to" goes on the bottom. This applies to men and women, and to the 3 and 1 (except the "of" is implied in "3 to 1"). The "is" still means "equals". So we get

$$\frac{\text{men}}{\text{women}} = \frac{3}{1}$$

or, since 3/1 is the same as 3,

$$\frac{\text{men}}{\text{women}} = 3$$

Example 2:

For every six magazines he buys, Ed buys one book.

This tells you that Ed buys six times as many magazines as books. We could write this as

$$\text{magazines} = 6(\text{books})$$

or, to make it look more like a ratio

$$\frac{\text{magazines}}{\text{books}} = 6$$

If Ed buys six magazines, he buys 1 book. If he buys 12 magazines, he buys 2 books, 18 magazines, 3 books, etc. The words "for every" tell you to construct a ratio. The sentence will say something like "for every X"; the X is one of the two types of things that will be in the ratio. The X can go on the top or bottom of the ratio, as long as the number associated with the X goes on the corresponding part of the fraction. Here, for example, we have "for

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[1] There is a reason why ratios are called proportions. The "pro" means "for". In a ratio, we have a certain portion (amount) of one thing *for* (pro) every one of another thing.

every *six* magazines…" We put magazines on the top of our ratio, which means that the six has to go on the top of the fraction (6/1). But, we could have put magazines on the bottom, as long as the 6 went on the bottom of the fraction. We would get

$$\frac{\text{books}}{\text{magazines}} = \frac{1}{6}$$

This is the same ratio as the one given above.

Example 3:

Superbars contain 10 grams of protein per gram of fat.

The word "per" means the same as "for every"; the only difference is that whatever comes after the "per" goes on the bottom of the ratio. We have to assign the numbers accordingly; the 10 goes with protein, and the implied 1 goes with fat ("per gram of fat" means "per *one* gram of fat.").

$$\frac{\text{protein}}{\text{fat}} = 10$$

Example 4:

For every 10 people, 6 don't like American cheese. What is the ratio of people who like American cheese to those who don't?

Ok, here we have our first actual question. This is an important example, so pay attention.

The first thing you should realize is that this is a word problem. What does that mean? It means that you tackle it just like any other word problem: one sentence at a time. The first sentence involves a crucial concept in ratios: If you are told how many people (or things) *don't* do something, you know that the remainder *do* do it. If you are told how many people (or things) *do* something, you know that the rest *don't*. Here, we know that 6 out of 10 don't like American cheese. This means that the remaining 4 (we get the remainder by subtracting the one group from the total; 10 – 6 = 4) *do* like American cheese.

On the GMAT, you can often say that there are only two kinds of people – those who X and those who don't X. This is important! The minute you see a sentence about a part of a group who does something, you know the rest don't do it, and vice versa. Always figure these numbers out before you continue with the problem.

With that information in mind, we go on to the next sentence. We are asked for the ratio of those who like to those who don't like American cheese. That means that like is on the top and don't like on the bottom of the ratio.

$$\frac{\text{like}}{\text{don't}} = \frac{?}{?}$$

To construct the ratio, plug in the number you know. 4 like it and 6 don't.

$$\frac{\text{like}}{\text{don't}} = \frac{4}{6}$$

Always reduce the fraction.

$$\frac{\text{like}}{\text{don't}} = \frac{2}{3}$$

This question really contains a trap. Most people will use the 6 and the 10 in the ratio, rather than 6 and 4, and they'll get the wrong answer.

Example 5:

For every person who exercises regularly in the US, nine people don't. What is the ratio of regular exercisers to total people?

The first sentence tells us that there are two kinds of people in the US: regular exercisers, and non-regular exercisers. The words "for every" tell us to construct a ratio. I'll put regular exercisers on the top (it doesn't matter); there is an implied 1 associated with regular exercisers in the sentence ("for every person who exercises regularly" means "for every *one* person"). The 1 goes on top of the fraction, 9 on the bottom.

$$\frac{\text{regular}}{\text{non}} = \frac{1}{9}$$

Now we read the next sentence. It asks us for a ratio; the trap here is that it asks us for a different ratio than the one we've just constructed. It was the ratio of regular exercisers to *total people*.

$$\frac{\text{regular}}{\text{total}} = \frac{?}{?}$$

When we see "total" we think addition. What do we add up to get the total number of people? Well, there are only two types of people: regular exercisers, and non-regular exercisers. We have 1 regular and 9 non-regular (these aren't the actual total number of people in the US, but, as I'll show you soon, we can act as if they are for now). Add them together to get the total, because there are no other types of people. The total number of people is 10. The number of regular exercisers is 1. So

$$\frac{\text{regular}}{\text{total}} = \frac{1}{10}$$

What lesson are we learning from all this? Well, if we have a general group, we can always split it into two groups: those who have quality X, and those who don't have quality X. If we know how many X and how many don't X, we can figure out the total. One thing to be careful of, however, is that these numbers – the total, the number of people who X, etc – often aren't the actual number of each thing. For example, even if the ratio of regular exercisers to total people is correct, this *doesn't* mean that there are only 10 people in the United States. Ratios don't give you actual numbers, because they are always fractions which are reduced to their **lowest terms**. I will discuss this in more detail later, but I just want to make sure you understand that numbers you get and use in ratios are *related* to actual numbers, but are not always the actual numbers themselves.

> "Reduced to lowest terms" means that you have divided both top and bottom of the fraction by any number that will go into both. Look at the fraction 2/6 for example. 2 will divide into both the top and bottom, so this is not in lowest terms. If we divide out 2, we get 1/3, which is really the same number, just reduced properly. See **Fractions** in the **Appendix ????** for more information.

Now, let's get more of a taste for how ratios can be used in GMAT questions.

Example 6:

> There are a total of 240 teachers and students in a school. The ratio of teachers to students is 1 to 5. How many teachers are there at the school?

The rules for word problems still apply, so we deal with this sentence by sentence. The first sentence uses the word "total", so we are going to add things to get this 240. What do we add? We add the number of teachers and the number of students. Create an equation, and remember to label your variables.

t = number of teachers
s = number of students

$$t + s = 240$$

Good. We can't solve this and get a value, so we move on.

The next sentence tells us about ratios. We are going to put the ratio in an equation. The "of" goes on top and the "to" on the bottom, so we get (using the same variables):

$$\frac{t}{s} = \frac{1}{5}$$

Now, can we solve for anything? Well, how many variables do we have? How many equations? That's right, we have two variables and two equations, so we can solve. This is why we write ratios as equations: so we can use them in solving other equations.

There are several ways we can solve this. We could solve the first equation for t or s and substitute that into the second equation. We can solve the second equation for s and substitute that into the first equation. Either of these is great. But I'm going to teach you a little "trick" that will often make ratios very easy to solve. This trick is called **cross multiplying**. Here's how it works. <u>When you have a simple ratio equation (one ratio equals another (or a ratio equals a fraction)) you multiply the numerator of each side by the denominator of the other</u>. Here, the numerators are t and 1, and the denominators are s and 5. We multiply t by 5 (the denominator of the *other* side) and s by 1 (the denominator of the *other* side) to get:

$$5t = s$$

Why does this work? Well, it works because what we are really doing is multiplying both sides by s and both sides by 5. The s cancels on the left side and the 5 cancels on the right; this ends up looking like we just multiplied each side by the denominator of the other. Now, we have solved for s very nicely. We *could* have just solved for t, by multiplying both sides by s, but we would have gotten (t = (1/5)s), which isn't as easy to use as what we have now. My rule of thumb is <u>always cross multiply when you want to use ratios in algebra</u>.

Now we substitute what we have into the first equation.

$$t + s = 240$$

$$t + 5t = 240$$

$$6t = 240$$

$$\frac{6t}{6} = \frac{240}{6}$$

$$t = 40$$

Now go on to read the question. It asks for the number of teachers, which we've already gotten. We're done.

Lets do another example.

Example 7:

If there are 30 more women than men in a group, and the ratio of men to women is 2 to 5, what is the ratio of women to total people in the group?

This problem is one sentence long, so rather than doing it sentence by sentence, we'll tackle it clause by clause. The first clause is a little weird. It is clearly an equation, but the "are" is in a weird place, so we don't know what equals what. We know that women are more than men. "More" means add, so we are adding 30 to something, but to what? Well, it is "more than men" so we add to "men" (just like if it were "less than x" we 'd subtract from x). We get

m = number of men
w = number of women

$$w = 30 + m$$

We can't solve, so we read on. This next part is also an equation. I'll let you construct it, don't look at what I have…

$$\frac{m}{w} = \frac{2}{5}$$

Did you look? OK, now what do we do? We've got two variables and two equation, so we'll solve. What's our rule of thumb? That's right, cross multiply the ratio. We get

$$5m = 2w$$

Not very helpful. *But*, the first equation is already solved for w, so we substitute that into the second equation. And, in fact, we are glad that we cross multiplied, because now we don't have any fractions to worry about.

$$5m = 2(30 + m)$$

$$5m = 60 + 2m$$

Go ahead and solve. Get the m's together first.

$$\begin{array}{rcl} 5m & = & 60 + 2m \\ -2m & & -\ 2m \\ \hline 3m & = & 60 \end{array}$$

$$\frac{3m}{3} = \frac{60}{3}$$

$$m = 20$$

Excellent. Now we keep reading. The question asks for the ratio of women to the total number of people. Write this down.

p = total number of people

$$\frac{w}{p} = \frac{?}{?}$$

We only know the number of men so far. We can use that to get the number of women. there are 30 more women than men, so we have 50 women. Now we need the number of people. Since that is a total, we add together the number of men and women, to get 70. Put this in our ratio:

$$\frac{w}{p} = \frac{50}{70}$$

Notice how important it is to do everything step by step. If you read the whole question straight through and attempt to solve it all at once, you are going to be slower because you'll be thinking about too many things at one time.

Now let's do some data sufficiency.

Example 8:

Javier and Sandy are playing checkers. Javier is red and Sandy is black. How many red pieces are on the board?

1) The ratio of Javier's remaining pieces to Sandy's is 5:2.
2) Sandy has 6 less pieces than Javier.

A. Statement (1), by itself, is sufficient to answer the question, but statement (2), by itself, is not.
B. Statement (2), by itself, is sufficient to answer the question, but statement (1), by itself, is not.
C. Statements (1) and (2) taken together are sufficient to answer the question, although neither statement by itself is sufficient.
D. Either statement by itself is sufficient to answer the question.
E. Statements (1) and (2) taken together are not sufficient to answer the question, nor are they sufficient to answer the question by themselves.

Let's create some variables before we look at the statements.

j = number of Javier's pieces
s = number of Sandy's pieces

$$j = ?$$

Statement 1 should be written as an equation.

$$\frac{j}{} = \frac{5}{}$$

s   2

Can we get a value for j? No. We have two variables (j and s) but only one equation.

Statement 2 is also an equation.

$$s = j - 6$$

This is also insufficient, because it is also only one equation. But, if we combine the two statements we have two equations; thus, they are sufficient together.

The Limitations of Ratios

I mentioned before that ratios are fractions which are always reduced to their lowest terms. This means that the numbers in ratios are not the same as actual numbers of things in the real world. To make that clearer, let's look at an example. Do you remember those old Trident gum commercials? They told you that 4 out of 5 dentists recommended Trident for their patients who chewed gum. This can be expressed as:

$$\frac{\text{dentists who recommend trident}}{\text{all dentists}} = \frac{4}{5}$$

Now, here's the question. Let's assume that this ratio is true. Does this mean that there are only 5 dentists in the whole world? Or that Trident only asked 5 dentists? Of course not. What is means is that there are 500 or 5000 or whatever dentists, and 4/5 of them recommended Trident. Notice that we *don't* know how many dentists there are; there could be almost *any* number (within certain limits which we'll get to soon), as long as 4/5 of them recommend Trident.

This has some repercussions for Data Sufficiency. It means that *just* knowing a ratio doesn't every give you the value of the numbers involved. We know the ratio of recommending dentists to all dentists, but we don't know how many of each type of dentist there really are. However, as long as you write ratios as equations, you don't have to remember that they don't give you values. This is because every ratio equation will contain two variables, so every ratio equation will be insufficient on its own. They always need another equation to make them sufficient.

Example 9:

How many married people are in a group?

1) The ratio of married people to unmarried people is 3 to 2.
2) If the number of married and unmarried people were tripled, there would be 9 married people for every 6 unmarried people.

A. Statement (1), by itself, is sufficient to answer the question, but statement (2), by itself, is not.
B. Statement (2), by itself, is sufficient to answer the question, but statement (1), by itself, is not.
C. Statements (1) and (2) taken together are sufficient to answer the question, although neither statement by itself is sufficient.
D. Either statement by itself is sufficient to answer the question.
E. Statements (1) and (2) taken together are not sufficient to answer the question, nor are they sufficient to answer the question by themselves.

Let's make a variable before we start.

m = number of married people

$m = ?$

Statement one gives us a ratio, which we should express as an equation.

u = number of unmarried people

$$\frac{m}{u} = \frac{3}{2}$$

This is insufficient, as we have two variables and one equation.

Statement two tells us that we are going to triple the married and unmarried people, which means that we have 3m and 3u. It also gives us an equation.

$$\frac{3m}{2u} = \frac{9}{6}$$

This is also insufficient, because it is only one equation.

Put them together: ahh, two variables, two equations, so this is sufficient, right? Wrong. Think back to **Example 20** in the **Algebra** chapter. We have one equation which is just a multiple of another. The equation in Statement 2 is just the equation in Statement 1 multiplied by 3. If we divided both sides by 3, we'd have the exact same equation. This doesn't give us two equations, just two versions of the same thing (if you don't believe me, try solving and substituting, see what happens). Thus, together they are still insufficient. Notice, the same skills we learned in **Algebra** are used in **Ratios**; like I said at the beginning, we want to re-use the same ideas over and over, rather than having to learn new stuff.

Example 10:

> The ratio of green to blue marbles in a bag is 3 to 4. In the same bag, the ratio of green to red marbles is 3 to 8. If there are no other color marbles in the bag what is the smallest number of marbles that the bag could contain?

OK, here's where we explain the real interesting stuff about ratios. Go through the first two sentences and construct your equations.

g = number of green marbles
b = number of blue marbles
r = number of red marbles

$$\frac{g}{b} = \frac{3}{4}$$

$$\frac{g}{r} = \frac{3}{8}$$

Now, we want to know the smallest number of marbles the bag can contain. There is a quick way to figure this out, and I'll show you what that is soon, but in order for it to really make sense, we have to go through a certain process. At the end of it, you'll truly understand ratios. Excited? Let's go!

If we want the smallest number of total marbles, we want to have the smallest number of each kind of marbles, right? Let's think about green marbles first – what is the smallest number of green marbles we can have? Well, the smallest number I can think of is 1. Can we have just 1 green marble?

In order to figure this out, assume we *had* 1 green marble. Plug that into our equations and solve for how many blue and green marbles we would have. So we replace g with 1 (we're assuming g = 1) and solve.

$$\frac{1}{b} = \frac{3}{4}$$

Cross multiply:

$$4 = 3b$$

$$\frac{4}{3} = \frac{3b}{3}$$

$$b = \frac{4}{3}$$

So if we had one green marble, we'd have 4/3 blue marbles. This is impossible: we can't have 4/3 of a marble.[2] So we can't have just one green marble. What's the next smallest number to try? What would happen if we had 2 green marbles? I'll let you handle the math.

If we had 2 green marbles, we would have 8/3 blue marbles. Again, this is impossible. Now try 3 green marbles (the next smallest number – remember, we are trying to get the smallest number of total marbles, which means that we want the smallest number of each kind of marble; we're going to get bigger and bigger until one works).

3 works – if we have 3 green marbles, we get 4 blue, and 8 red. Both of these are whole numbers. So 3 is the smallest number of green marbles, which means that 4 is the smallest number of blue marbles, and 8 is the smallest number of red marbles. The smallest number of total marbles is 15.

Notice something – we couldn't have a number of marbles less than the number given in the ratios. That is, we had to have at least 3 green marbles, and 3 is the number in the ratio corresponding to green. Likewise, we had to have at least 4 blue marbles, and 4 is the number corresponding to blue. I'm going to tweak the numbers in our example and show that we can get a useful rule out of this.

Example 10a:

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[2] I understand that we could actually split a marble into pieces and have just a little piece. But, on the GMAT, certain things can only come in whole numbers. For example, you can't have half a person, or half a car, etc.

> The ratio of green to blue marbles in a bag is 3 to 4. In the same bag, the ratio of green to red marbles is 4 to 5. If there are no other color marbles in the bag what is the smallest number of marbles that the bag could contain?

g = number of green marbles
b = number of blue marbles
r = number of red marbles

$$\frac{g}{b} = \frac{3}{4}$$

$$\frac{g}{r} = \frac{4}{5}$$

We don't need to go through entirely the same process as before. We know that we have to have at least 3 green marbles, based on what we did in **Example 10**. Actually, from the second equation, we know that we have to have at least *4* green marbles, because 4 is the number corresponding to green in the second ratio. The question is, can we really have 4 green marbles? Try 4 out for g in the first ratio.

$$\frac{4}{b} = \frac{3}{4}$$

Cross multiply:

$$16 = 3b$$

$$\frac{16}{3} = \frac{3b}{3}$$

$$b = \frac{16}{3}$$

This doesn't work – we don't get a whole number for b. I want you to work your way up through the numbers, one by one, until you get a number that can work for g in both the first and second equation. Notice which numbers work for g in the first, and which work in the second, and stop when you get to one that works in both.

OK, finished? You should notice that 3, 6 and 9 work for g in the first ratio, but not the second. And 4 and 8 work for g in the second, but not the first. 12 works in both. Do you notice a pattern? In the first equation, the number in the ratio corresponding to g is 3, and 3 goes into each number that works. In the second ratio, 4 corresponds to g, and 4 goes into each number that works. 12 is the smallest number that both 3 and 4 go into, so 12 is the smallest number that works in both equations.

Here is the rule: <u>when you have a ratio reduced to its lowest terms, each value in the ratio has to be a **multiple** of the number corresponding to it</u>. So in the ratio

$$\frac{g}{b} = \frac{3}{4}$$

g has to be a multiple of 3 (3, 6, 9, 12…) and b has to be a multiple of 4 (4, 8, 12 …).

A **multiple** of a number x is x times some number (hence "multiple" because we multiply x by something). Any multiple of some number can be **divided evenly** by that number; that is, we can divide them and not get a remainder. Multiples of 3, for example (3, 6, 9, 12, 15…) can all be divided by 3 and we won't get a remainder.

There is a reason for this. If one of the terms is *not* a multiple of the right number, the other term will come out to be a fraction. If b is 6, not 4 or 8, g will have to be a fraction. Why is this? Well, let's cross multiply the ratio and look at it that way.

$$4g = 3b$$

$$g = \frac{3b}{4}$$

So g is 3/4 times b. Whatever b is, you divide it by 4 (and multiply by 3) to get g. If g is not a multiple of 4 (that is, if it can't be divided evenly by 4), then g will have to be a fraction, because it will have that 4 in the denominator (the 4 only cancels out if it can divide into the b). Look at the equation another way:

$$b = \frac{4g}{3}$$

So g has to be a multiple of 3, because we are going to divide it by 3 to get b. If b *isn't* a multiple of 3, then b will be a fraction, because the 3 won't divide into it. This is true for every single ratio no matter what, and we could have used this information to solve the problem quickly. g is in both ratios; we know, from the two ratios, that g has to be a multiple of 3 and a multiple of 4; the smallest number that is a multiple of both is 12. So the smallest g can be is 12.

If g is 12, what are b and r? Go ahead and solve for each of them, I'll wait.

You should have gotten 16 for b and 15 for r. Now, I want you to notice something. In the first ratio, 3 corresponds to g. g is actually 12, and 12 is 3 times 4. Notice that b ends up being 16, which is 4 times 4. In the second equation, 4 corresponds to g; 4 goes into 12 three times. r ends up being 15, which is 5 times 3. There a rule for this: <u>whatever you multiply one part of the ratio by to get the real number, you multiply the other part by the same amount</u>. For example, if g was actually 24, b would have to be 32. This is because we multiply 3 (the number corresponding to g) by 8 to get 24, so we multiply 4 (the number corresponding to b) by 8 as well. This should make sense: if we double the top of a ratio, we have to double the bottom as well, or the fraction won't work out any more. So, <u>in a ratio, if we multiply one part, we have to multiply the other part by the same amount</u>.

Notice also that in these ratios we can create a ratio of one type of marble to the total number of marbles. For example, in **Example 10**, for every 3 green marbles there are 15 marbles in total (because there will be 4 blue and 8 red as well). We can make ratios from this:

$$\frac{g}{total} = \frac{3}{15} = \frac{1}{5}$$

$$\frac{b}{total} = \frac{4}{15}$$

$$\frac{r}{\ } = \frac{8}{\ }$$

total 15

We can use these ratios just like any other ratios. That is, we know that if we multiply one part of the ratio, we have to multiply the other part by the same amount. Let's look at an example of how we would use this.

Example 10b:

Using the same ratios of green, red and blue marbles as in **Example 10**, (i.e., green to blue is 3 to 4, green to red is 3 to 8) if there are 60 marbles in total, how many blue marbles would there be?

We could solve this algebraically – set up the ratio of b to total, plug into 60 for total, and solve for b. But, we can also notice that the total number of marbles has to be a multiple of 15. 60 is 4 times 15; thus we know that we have multiplied the ratio by 4. This means that we have to multiply every part of the ratio by 4. Thus, we have to have 16 blue marbles.

Take a look at these next two practice questions. Use what you have just learned to answer them, then read my explanations.

Practice Question Set 1

1. Given the circumstances described in **Example 10a**, what is the smallest number of blue marbles?

2. Given the circumstances described in **Example 10a**, if there were 129 marbles total, how many of them would be green? How many would be red?

Explanations for Practice Question Set 1

1. Here are our equations (from **Example 10a**):

g = number of green marbles
b = number of blue marbles
r = number of red marbles

$$\frac{g}{b} = \frac{3}{4}$$

$$\frac{g}{r} = \frac{4}{5}$$

You might be tempted to say that the smallest number of blue marbles is 4; after all, that's the number given for b in the ratio. But this doesn't work: if there are 4 blue marbles, there are 3 green. But, if you look at the second ratio, there have to be a minimum of 4 green.

You can figure out that there have to be a minimum of 12 green marbles (see **Example 10a**). This means that the first ratio, the one containing b, is multiplied by 4. Thus, we have to multiply both g and b by 4. The minimum number of blue marbles is 16.

2. This question talks about the total number of marbles, so we should construct ratios comparing the number of blues, greens, and reds to the total number of marbles. That requires us to figure out what the minimum number of each type of marble is, so that we can figure out the total. We just saw that there have to be 12 green and 16 blue. There

have to be 15 red, since we multiply the second ratio by 3 to get 12 green. In total, then, there are at least 43 marbles. Let's make some ratios (remember to make the ratio between the minimum number of each type of marble and the minimum number of total marbles):

t = total number of marbles

$$\frac{b}{t} = \frac{16}{43}$$

$$\frac{g}{t} = \frac{12}{43}$$

$$\frac{r}{t} = \frac{15}{43}$$

OK, now we know that we actually have 129 marbles in total. We can get the number of greens and reds by substituting 129 into each equation and solving, or by asking how many times do we have to multiply the minimum total to get 129? You can try it the first way by yourself. You can quickly see (through long division) that 129 is 43 times 3. So we triple each ratio. This means that we have 48 blue, 36 green, and 45 red.

Put It All Together

Now I'm going to give you some more practice questions. Each involves ratios and "regular" equations. Use your algebra skills, and try using what we've just learned about ratios. Two of the questions involve ratios between three things; these can be treated as two separate ratios (just as we did in **Example 10** and **10a**).

Practice Question Set 2

1. At a circus, the ratio of monkeys, clowns and trapeze artists is 5 to 7 to 8. If these are the only types of performers at the circus, and there are 120 performers total, how many are clowns?

2. What is the value of x?

   1) The ratio x:y:z is 1:2:3.
   2) y is 4 times the value of the number which is 1 one greater than z.

   A. Statement (1), by itself, is sufficient to answer the question, but statement (2), by itself, is not.
   B. Statement (2), by itself, is sufficient to answer the question, but statement (1), by itself, is not.
   C. Statements (1) and (2) taken together are sufficient to answer the question, although neither statement by itself is sufficient.
   D. Either statement by itself is sufficient to answer the question.
   E. Statements (1) and (2) taken together are not sufficient to answer the question, nor are they sufficient to answer the question by themselves.

3. How many people of above average height belong to a certain club?

   1) The ratio of people of above average height to those whose height is average or below average is 2 to 3.
   2) There are 60 people in the club.

   A. Statement (1), by itself, is sufficient to answer the question, but statement (2), by itself, is not.
   B. Statement (2), by itself, is sufficient to answer the question, but statement (1), by itself, is not.
   C. Statements (1) and (2) taken together are sufficient to answer the question, although neither statement by itself is sufficient.
   D. Either statement by itself is sufficient to answer the question.
   E. Statements (1) and (2) taken together are not sufficient to answer the question, nor are they sufficient to answer the question by themselves.

Explanations Practice Question Set 2

1. Write down what the first sentence gives you as two different ratios. Remember to label your variables.

m = number of monkeys
c = number of clowns

t = number of trapeze artists

$$\frac{m}{c} = \frac{5}{7}$$

$$\frac{c}{t} = \frac{7}{8}$$

We could add one more ratio (m/t = 5/8) but that would be redundant – it wouldn't tell us anything we don't already know, because it is just substituting the first ratio into the second (try and see). We have three variables but only two equations, so we can't solve yet.

The next sentence tells us that we have 120 performers in total. We can write this as:

$$m + c + t = 120$$

At this point we could just solve algebraically (solve the two ratios for m and c, and substitute into the last equation). Try it if you like. It is easier, however, to use what we know about ratios. We can construct a ratio of clowns to the total number of performers. We know that there are 7 clowns minimum, because each ratio contains 7 for the number of clowns. This means that there are a minimum of 5 monkeys and 8 trapeze artists. From this we know that there are a minimum of 20 performers.

$$\frac{c}{total} = \frac{7}{20}$$

We actually have 120 performers in total, so we have to multiply this ratio by 6. This means that we have 42 clowns.

2. Write down what you are asked:

$$x = ?$$

Statement one gives us a three part ratio. Write it as two ratios:

$$\frac{x}{y} = \frac{1}{2}$$

$$\frac{y}{z} = \frac{2}{3}$$

We could write a third ratio (x/z = 1/3); this would give us three equations. But the third ratio would be redundant; it would just be putting the two ratios together. This means that it wouldn't could as a third *new* equation, so we couldn't say that we had three variables and three equations.

At this point we have only two equations, but three variables. We can't get a value for x.

Statement 2 gives us an equation. The number which is 1 greater than z is just (z + 1). So it says:

$$y = 4(z + 1)$$

This, by itself, tells us only about y and z, not x. This is insufficient. Be careful: it is easy to quickly say that Statement 2 is *sufficient*, because you are using what you already know from Statement 1. You have to be careful to keep the statements separate in your head unless they are both insufficient.

When we put the two statements together we see that we have three variables and three equations, so this is sufficient. The answer is C.

3. Assign a variable before looking at the statements.

a = number of people of average height

$$a = ?$$

Statement 1 gives us a new variable and an equation:

n = number of people not of average height

$$\frac{a}{n} = \frac{2}{3}$$

Notice that a and n together include all the people; there are no people who are not either of above average height or not of above average height.

This is insufficient: we can't solve this equation to get a number value for a.

Statement 2 gives us a total number of people in the club. We can write this as an equation:

$$a + b = 60$$

But, since we have to ignore the equation from statement 1, this is insufficient by itself. Again, putting them together they are sufficient.

Ratios and Percents